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Logical Reformulation of Quantum Mechanics. III. Classical Limit and Irreversibility

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This paper deals with two questions: (1) It contains a proof of the fact that consistent quantum representations of logic tend to the classical representation of logic when Planck's constant tends to zero. This result is obtained by using the microlocal analysis of partial differential equations and the Weyl calculus, which turn out to be the proper mathematical framework for this type of problems. (2) The analysis of the limitations of this proof turn out to be of physical significance, in particular when one considers quantum systems having for their classical version a Kolmogorov K-system. These limitations are used to show the existence of a "best" classical description for such a system leading to an objective definition of entropy. It is shown that in such a description the approach to equilibrium is strictly reduced to a Markov process.

KEY WORDS: Quantum mechanics; classical limit; microanalysis; irreversibility.

1. INTRODUCTION

This is the third and last of a series of articles[1] (the first two will be called I and II), that are concerned with the logical foundations of quantum and classical physics. In I the notion of a quantum representation of logic was introduced and one of the main results was to show how the classical representation of logic to which we are used arises as a limit of quantum representations when Planck's constant tends to zero.

However, the arguments given in I remained at an intuitive level and the main task of the present paper will be to offer a proof for them. It turns out that the proper framework to be used is provided by microlocal analysis, one of the parts of mathematics that had a most significant

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advance during the last two decades. In fact, microlocal analysis replaces the study of a linear partial differential equation such as the Schrödinger equation by a calculus in phase space, so that it deals particularly well with the localization properties of Fourier transforms. As such, it is perfectly suited to the kind of question of interest here. I have not been able to find proofs resting only upon elementary analysis, so I chose rather to apply directly the methods and the results given by Hörmander.[2]

A theory of measurement was also proposed in I, where of course the question of irreversibility was met. The present analysis of the conditions where quantum logic tends to classical logic exhibits a significant exception when the quantum system under consideration has for its classical version a Kolmogorov mixing K-system. Furthermore, the microlocal analysis of classical mechanics made by Fefferman,[3] where classical mechanics was conceived as an approximation to quantum mechanics up to a given *relative* error, allows one to characterize what may be the "best" limiting description of a system in the proposition calculus, i.e., the "most" one can tell consistently about the system.

These two remarks, when taken together, strongly suggest a new approach to the foundations of thermodynamics that remains in the general framework I have just been constructing, so far as it may offer an objective definition for the "best" description of a physical system by a classical distribution function. It is shown that the master equation for such a distribution reduces completely the problem of the approach to equilibrium to the study of a Markov process.

This question of irreversibility somewhat exceeds the limits of the present investigation, so that it will be presented here only in general terms and more as a program to be developed independently.

2. W-SYMBOLS AND OPERATORS

I shall consider a quantum system consisting of a finite number of interacting particles, the total number of degrees of freedom, i.e., thrice the number of particles, being denoted by N. The position and momentum coordinates will be denoted collectively by $x=(x_1,...,x_N)$ and $p=(p_1,...,p_N)$. I shall use a scalar product

$$p\cdot x=\sum_{j=1}^{N} p_j x_j$$

I shall consider square integrable wave functions $u(x)$ as well as their Fourier transforms $\tilde{u}(p)$ in the form

$$u(x) = \int h^{-N}\, dp \exp(ip \cdot x/h)\ \tilde{u}(p)$$

$$\tilde{u}(p) = \int dx \exp(-p \cdot x/h)\, u(x)$$

To an operator A, I shall associate a Wigner–Weyl function, or W-function for short, defined by[4,5]

$$a(x, p) = \int \langle x' | A | x'' \rangle\, \delta[x - (x' + x'')/2] \exp[-ip \cdot (x' - x'')/\hbar]\, dx'\, dx'' \tag{2.1}$$

I shall use systematically the same lower-case or capital letter for the W-function or the operator. When $a(x, p)$ is an indefinitely differentiable (C^∞) function and when furthermore its derivatives are bounded in the form

$$|(\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_N)^{\alpha_N} (\partial/\partial p_1)^{\beta_1} \cdots (\partial/\partial p_N)^{\beta_N} a(x, p)| \leqslant C_{\alpha\beta}(p_0^2 + p^2)^{(m - |\beta|)/2} \tag{2.2}$$

where $\alpha_1, \ldots, \beta_N$ are integers, $|\beta| = \beta_1 + \cdots + \beta_N$, and $C_{\alpha\beta}$ is a constant, m a positive or negative number, it is said that $a(x, p)$ is a W-symbol of order m. For instance, $a(x, p) = x$ defines a symbol of order 0 and $a(x, p) = p$ a symbol of order 1. Note that the functions x and p are respectively the symbols of the operators X and P for position and momentum.

I shall use the following properties involving the norm and the trace:

$$\|A\| \leqslant \int |a(x, p)|\, dx\, dp\, h^{-N} \tag{2.3}$$

$$\operatorname{Tr} A = \int a(x, p)\, dx\, dp\, h^{-N} \tag{2.4}$$

To the adjoint A^+ of an operator is associated the complex conjugate W-function $a^*(x, p)$, so that a real W-function is associated with a self-adjoint operator. When A is a positive-definite operator, its W-function is not necessarily positive everywhere. This is well known in the association between a density operator and a Wigner statistical distribution function in (x, p) space.[5] Conversely, the operator associated with a positive W-function does not necessarily have a spectrum that is confined to the positive real axis.

The product of two operators, when written in terms of the associated W-functions, is not trivial. In order to express it, it will be convenient first

to define the Poisson bracket as an operator acting between two functions in the following way: denote by $\{\cdot\}$ the operator

$$\{\cdot\} = \partial/\partial x \cdot \partial/\partial p - \partial/\partial p \cdot \partial/\partial x \tag{2.5}$$

where in each scalar product of derivation operators, the first derivative acts on the function standing on the left and the second one acts on the right. For instance,

$$a\{\cdot\}b = \{a, b\} \equiv \partial a/\partial x \cdot \partial b/\partial p - \partial a/\partial p \cdot \partial b/\partial x$$

To the operator $C = AB$ is associated the W-function

$$c(x, p) = a \exp(-i\hbar/2\{\cdot\})b \tag{2.6}$$

which can be expressed more explicitly by using Fourier transforms as

$$c(x, p) = (h/2)^{-2N} \int a(x+y, p+q)\, b(x+z, p+s) \times \exp[2i(s \cdot y - q \cdot z)/\hbar]\, dy\, dz\, dq\, ds \tag{2.7}$$

When a and b are W-symbols, c is also a W-symbol.

I can now describe the general idea of the present approach: It consists in associating a *quantum* projector E with the *classical* proposition "The system is in a domain D of phase space." A natural idea would be to take for the W-function associated with the operator E the characteristic function of D that is equal to one in D and to zero outside. However, it is a discontinuous function and does not give rise to a good projector. To do so, one must use a C^∞ function. Conversely, it will be found that such a smooth function approaching the characteristic function of D generates a projector when D satisfies some necessary conditions. This leads us to introduce what is called in the literature a Schwartz function.

3. SCHWARTZ FUNCTIONS

This short section will be devoted to a few technical details about Schwartz indefinitely derivable functions and how they can be used to smooth the boundary of a domain in phase space.

Let us consider a C^∞ function of one variable $F(x)$ that is everywhere equal to zero except in the interval $[0, 1]$, where it is positive. It vanishes together with all its derivatives when x is equal to 0 or to 1. I shall assume that its integral over $[0, 1]$ is equal to unity.

I shall also consider the primitive function

$$G(x) = \int_x^\infty F(t)\, dt, \qquad x \geqslant 0$$

$$= 0, \qquad x < 0$$

It is a C^∞ function except at $x = 0$; it is identical to zero for $x < 0$ and $x \geqslant 1$. It is discontinuous at $x = 1$, where its value passes abrupty from 0 to 1 when x crosses zero by increasing values.

It will be necessary to know what bounds can be put on such functions and on their derivatives. According to a theorem by Denjoy and Carleman (see ref. 2, Vol. 1), the best one can do in order to get the smallest bounds is the following: Take an infinite sequence of positive numbers $a_1, a_2, ..., a_n, ...$ such that their sum is equal to one. Then, it is possible to have the bounds

$$|G^{(k)}(x)| \leqslant 2^{k-1}[a_1 a_2 \cdots a_k|^{-1} = C_k, \qquad k > 1 \tag{3.1}$$

$$G(x) \leqslant 1 = C_0 \tag{3.2}$$

These inequalities show that one can bound G and its derivatives by numbers that are of the oder of unity if the order of derivation k is kept small enough. However, when we let k become large, the right-hand side of inequality (3.1) grows more rapidly than $k!$ times a quantity increasing itself more rapidly than the kth power of any given number. This very rapid increase will turn out to be a serious hindrance in considering macroscopic systems because then one will have to deal with values of k that will be larger than the number of degrees of freedom.

I now introduce a few useful notions relating to domains in phase space.

Let D be a simple domain in phase space (i.e., connected and simply connected). Call B its boundary, assumed to be C^∞ manifold. In a small neighborhood of a small region of B, one can use local coordinates $(z, u_1, ..., u_{2N-1})$, where z is a distance to B, counted positively outside D. Let λ be a given number and call the *margin* M of D the region where z is positive and smaller than λ. For the margin to be completely defined, it would be necessary to make the local coordinates more precise, but this will come later.

Define a function by its expression in local coordinates:

$$\psi(x, p) = G(z/\lambda) \tag{3.3}$$

It is a C^∞ function, with support given by M, it is equal to one on B, and it vanishes together with all of its derivatives on the outside boundary of

M. Let $\chi(x, p)$ denote the characteristic function of D, which is equal to 1 in D and 0 outside. The function

$$\phi(x, p) = \chi(x, p) + \psi(x, p) \tag{3.4}$$

is everywhere C^∞, is equal to one in D, and vanishes along with all its derivatives on the outside boundary of M. Its support is $D \cup M$.

4. ESTIMATES FOR AN OPERATOR PRODUCT

Let us now consider two domains D_1 and D_2 in phase space having boundaries B_1 and B_2 and margins M_1 and M_2. I introduce the two smoothed characteristic functions ϕ_1 and ϕ_2 as defined above as well as the two operators Φ_1 and Φ_2 admitting them as their W-symbols. It will turn out later that these operators approximate some projectors that will be used to define the limit of quantum logic to classical logic. Their commutators and therefore their products will be of interest.

Denote the volume of a region D in phase space by

$$[D] = \int_D dx\, dp \tag{4.1}$$

I shall prove the following theorems.

Theorem 1. For $[D]$ large enough, when the boundary manifold B is regular enough and the number of degrees of freedom is small enough, the operator Φ is approximately equal to a projector in the Hilbert space $\mathscr{H}$ of square integrable wave functions.

Theorem 2. For two fixed regions D_1 and D_2 of phase space obeying the conditions of Theorem 1, the commutator $[\Phi_1, \Phi_2]$ of the corresponding approximate projectors tends to zero in norm when $\hbar$ tends to zero.

The wording of these theorems remains somewhat vague but the exact limitations will appear in the course of the proof.

Clearly, Theorem 1 provides a wide family of projectors as were used in I to define quantum predicates, whereas Theorem 2 is the essential tool to prove that quantum representations of logic tend to the classical representation when Planck's constant tends to zero. Furthermore, it will be found that the limitations such as "regular enough" and "small enough" have important physical significance.

Theorem 1 is given without proof by Fefferman.[3] The proof of both theorems is an elementary exercise in microlocal analysis if one uses the results contained in Hörmander's book (ref. 2, Vol. 3), Chapter 18).

Everything relies on a bound that can be imposed on the quantity

$$u_{12}(x, p) = \phi_1 \exp[-i\hbar\{\cdot\}/2]\phi_2 - \phi_2 + i\hbar/2\{\phi_1, \phi_2\} \tag{4.2}$$

which will be used as such in the proof of Theorem 2 and with $\phi_1 = \phi_2 = \phi$ in the proof of Theorem 1. The quantity u_{12} represents the rest of the W-symbol associated with the product $\Phi_1 \Phi_2$ when the first two terms of Eq. (2.6) have been subtracted.

The proof is obtained in three steps.

1. Localization. I shall restrict attention to the case where both regions D_1 and D_2 have essentially the same size L in the x direction and P in the p direction. By this I mean that, up to some numerical factor of order unity, all the dimensions of these domains are comparable to L or P.

Introduce a neighborhood V of $D_1 \cup D_2$ and cover it by several cells C_α that may have curved boundaries. The precise definition of such a cell is the following: A canonical transformation v: $(x, p) \to (x', p')$ can bring it upon a rectangular cell in the variables (x', p'). It will be convenient to limit furthermore the domains under consideration by assuming that, for each index α, the intersection $B_1 \cap B_2 \cap C_\alpha$ is either a manifold $B_{12\alpha}$ of dimension $2N-2$ along which B_1 and B_2 are transverse or that it has dimension $2N-1$ and then coincides with both $B_1 \cap C_\alpha$ and $B_2 \cap C_\alpha$. Recall that two manifolds are said to be transverse when they can be written as $f(x, p) = 0$, $g(x, p) = 0$, and when their "normals"

$$n = (t, \tau) = (\partial f/dx_1, ..., \partial f/dx_N; \partial f/\partial p_1, ..., \partial f/\partial p_N)$$

and $n' = (z, \zeta)$ satisfy the condition $\sigma(n, n') \neq 0$, where σ is the symplectic form

$$\sigma(t, \tau; z, \zeta) = (\tau \cdot z - \zeta \cdot t)/\hbar \tag{4.3}$$

In the first case, $B_1 \cap C_\alpha$ will be brought upon the plane $x_1' = 0$ by the canonical transformation v and $B_2 \cap C_\alpha$ on the plane $p_1' = 0$. In the second case, $B1 \cap B_2 \cap C_\alpha$ will be brought upon the plane $x_1' = 0$.

Let ν denote the number of the cells covering V.

Since the Poisson parenthesis operator is invariant under canonical transformations, the proof of inequality (4.2) can be localized by considering it in each cell C_α and working with the transformed rectangular boxes C_α' in the coordinates (x', p'). More precisely, it will be assumed that the matrix elements of the Jacobian for the canonical transformations v^α are uniformly bounded by a number of the order of one once the units L and P have been chosen. These conditions imply that a region having a complicated boundary will have to be covered by a large number of cells ν.

I now specify the margins. In the case of intersection, the local margin of D_1 in C'_α is taken to lie between the planes $x'_1 = 0$ and $x'_1 = L\Delta$, and the margin of D_2 between $p'_1 = 0$ and $p'_1 = P\Delta$, with Δ being a small number fixed once and for all and taken to be the some in all the different cells.

2. Metrization. Using Hörmander's method (ref. 2, Chapter 18.5), I introduce a metric in C'_α that will be given by

$$g_{x',p'}(dy, dq) = (L^2 + x'^2)^{-1}\, dy^2 + (P^2 + p'^2)^{-1}\, dq^2 \tag{4.4}$$

The dual metric with respect to the symplectic form (4.2) is then given by

$$\begin{aligned} g^\sigma_{x',p'}(t, q) &= \sup_{y,s} [\sigma(t, q; y, s)]^2 / g_{x',p'}(y, s) \\ &= \hbar^{-2}[(P^2 + p'^2)t^2 + (L^2 + x'^2)q^2] \end{aligned} \tag{4.5}$$

The corresponding Hörmander bounding function is then given by

$$\begin{aligned} h(x', p') &= \sup_{t,q} q_{x',p'}(t, q)/g^\sigma_{x',p'}(t, q) \\ &= \hbar[(L^2 + x'^2)(P^2 + p'^2)]^{-1/2} \end{aligned} \tag{4.6}$$

3. Bounds. I now define a quantity $\|\phi\|_k$ that is a seminorm on the product $\phi_1\phi_2$: Denote collectively by X the coordinates (x', p'). The derivatives of order m of ϕ_1 over these variables generate a symmetric multilinear form $\phi_1(X'; T_1, \ldots, T_m)$ over the vectors tangent to phase space at a given point. The seminorm $\|\phi\|_k$ will be defined as the supremum of

$$\begin{Bmatrix} k \\ m \end{Bmatrix} |\phi_1(X'; T_1, \ldots, T_m)\, \phi_2(X'; T'_1, \ldots, T'_{k-m})| \left[\prod_{j=1}^{m} g_{X'}(T_j) \prod_{j=1}^{k-m} g_{X'}(T'_j) \right]^{-1/2}$$

In practice, it is a numerical constant depending upon the bounding constants C_r of the Schwartz functions up to order k times a power Δ^{-k} coming from the derivation because of our choice for the smoothing functions ψ and for the margins. I simply write this bound as

$$\|\phi\|_k = C'_k \Delta^{-k} \tag{4.7}$$

Then, according to Hörmander's Theorem 18.5.4, one has a bound for u_{12} that is of the form

$$|u_{12}(X')| \leqslant C'' h^n(X') \tag{4.8}$$

where C'' is a constant that is a linear combination of the constants C'_k up

to a value of $k = N + n + 2$, with n being an arbitrary positive integer that can be chosen at will. Therefore one has

$$|u_{12}(x', p')| \leqslant Ch^n(x', p')\, \Delta^{-(N+n+2)} \tag{4.9}$$

where C is again a numerical constant.

One can now use the inequality (2.3) to get a bound for the norm of the operator U_{12} associated with the W-symbol u_{12}. The integration over phase space appearing in Eq. (2.2) is performed separately in each cell C_α. Using the fact that

$$\int_{R^{2N}} dx'\, dp'\, h^n(x', p') = (c_n)^N (LP)^N h^n(0, 0) \tag{4.10}$$

where

$$c_n = \int_{-\infty}^{+\infty} du\, (1 + u^2)^{-n/2}$$

the resulting bound is finally

$$\|U_{12}\| \leqslant \nu C (c_n)^N (\hbar/LP)^{n-N} \Delta^{-(n+N+2)} \tag{4.11}$$

Proof of Theorem 1. Take $\phi_1 = \phi_2 = \phi$ and call Φ the operator having ϕ for its W-symbol. The operator Φ^2 has for its W-symbol

$$\phi^{(2)}(x, p) = \phi \exp(-i\hbar/2\{\cdot\})\phi \tag{4.12}$$

Introducing the difference

$$\rho_1(x, p) = \phi^{(2)} - \phi^2 \tag{4.13}$$

and taking into account the fact that $\{\phi, \phi\} = 0$, one can find that the operator R_1 associated with the W-symbol ρ_1 is, according to (4.11), bounded in norm by

$$\|R_1\| \leqslant \nu C (c_n)^N (\hbar/LP)^{n-N} \Delta^{-(n+N+2)} \tag{4.14}$$

On the other hand, from Eq. (3.3)

$$\phi^2 - \phi = \psi^2 - \psi \equiv \rho_2(x, p) \tag{4.15}$$

where, according to Eqs. (2.3) and (2.4),

$$\operatorname{Tr}|R_2| \leqslant \|R_2\| \leqslant [M] h^{-N} \tag{4.16}$$

The volume $[M]$ of the margin is of order $(LP)^N \Delta$. Since, according to Eq. (2.4), one has

$$\mathrm{Tr}\, \Phi = [D] h^{-N} \tag{4.17}$$

one gets

$$\Phi^2 - \Phi = R_1 + R_2 \equiv R \tag{4.18}$$

The operator R is compact, so that its spectrum is discrete with eigenvalues $r_1, r_2, \ldots$, satisfying, due to Eq. (4.16), the bound

$$\sum |r_j| \leqslant [M] h^{-N}$$

Furthermore, because of the property $0 \leqslant \psi - \psi^2 \leqslant 1$, all these eigenvalues lie essentially between 0 and 1 as a consequence of the sharp Gärding inequality.

As a result, Φ differs from a projector E associated with a subspace of $\mathscr{H}$ having dimension $[D] \cdot h^{-N}$ by a compact operator that has essentially the much smaller number $[M] \cdot h^{-N}$ of eigenvalues, all essentially contained in the interval [0, 1]. This is the exact meaning of Theorem 1.

Proof of Theorem 2. Theorem 2 is a simple consequence of the bound (4.11), since it implies

$$\| \Phi_1 \cdot \Phi_2 - i\hbar/2 \Phi_{12} \| \to 0$$

when $h \to 0$, Φ_{12} being the operator with symbol $\{\phi_1, \phi_2\}$ that is bounded. Interverting the functions given

$$\| \Phi_2 \cdot \Phi_1 + i\hbar/2 \Phi_{12} \| \to 0$$

so that $\| [\Phi_1, \Phi_2] \|$ tends to zero with h as stated by Theorem 2.

Addendum. It has been pointed out by L. Hörmander (personal communication) that these theorems were partly known.[11] Furthermore, he pointed out the following weaknesses in the above proof: (1) The operators must belong to the trace class (this is true for the applications to be used here). (2) The arbitrary power of h in formula (4.8) only occurs when one performs an expansion of $\phi^1 \exp[-ih/2] \phi_2$ to a given order that must be in general higher than the first order in h. In the present case, the known results[11] given a bound R as in Eq. (4.18) of the form $[M] h^{-N}$, as stated above, provided that $\Delta \geqslant h^{1/3}$.

5. PHYSICAL CONSEQUENCES

The preceding theorems have several useful physical consequences.

1. Theorem 1 gives us much more freedom when defining the predicates of a quantum representation of logic. When dealing with semiclassical statements that correspond to large volumes in phase space, we can agree to replace the projectors by the above operators $\Phi(D)$ without finding any practical difference. Then, using the freedom we have in choosing the smoothing functions, there is no difficulty in satisfying the compatibility conditions.

When dealing with predicates of the spcial type that were used in papers I and II, i.e., those associated with a part of the spectrum of a dynamical variable, it is possible to let the margins tend to zero in the following case. Let $F(p, q)$ be the dynamical variable; assume that there is a global canonical transformation transforming the function $F(p, q)$ into the variable x_1'; finally, let the domain D under consideration in the predicate be given by $a \leqslant x_1' \leqslant b$. Then the function ϕ associated to the domain D by Theorem 1 satisfies trivially the relation

$$\phi \exp(-i\hbar/2\{\cdot\})\phi = \phi^2$$

so that, when the margin becomes very small, one gets $\Phi^2 = \Phi$, which is the characteristic property of a projector. In particular, this is true for the dynamical variables p or q, the only operators to have been considered in II.

When we associate the operator Φ to a domain D in phase space to define a predicate, the complementary predicate is simply defined by the operator $I - \Phi$, which has practically the same properties with regard, to the domain complementary to D. More generally, one can define predicates corresponding to a Griffiths family by using a C^∞ partition of unity on phase space.

2. Theorem 2 provides all that is necessary to implement the intuitive considerations that were given in I for one of the most significant results of the present investigations, namely that *consistent quantum representations of logic tend to the classical representation when Planck's constant tends to zero.*

When dealing with the operators associated with a region D of phase space, this result assumes that the limitations that were found during the proof of the theorems are respected. These limitations will be discussed more carefully presently from a physical point of view.

3. In practice, h remains finite, so that the preceding results only hold approximately for regions of phase space having a large volume in units

h^{-N}. There is, however, a significant limitation when one considers macroscopic objects having a large number of degrees of freedom.

It has been noticed in Section 3 that the bounding constants increase very rapidly with the order of the derivation k and it was also found in Section 4 that one had to consider larger and larger orders of derivation when N increased. This may have the effect that any small relative value of h might be amplified tremendously in the bounds by the numerical coefficients when N becomes large.

In fact, this is no surprise and it should be so, since we know that many degrees of freedom remain quantized in a macroscopic object. The classical representation of logic only applies to collective variables, such as the motion of the center of mass. In fact, classical logic can go very deep in the description of matter, since we know, for instance, from the Born–Oppenheimer approximation, that it applies to the position of the atomic nuclei in a solid.

4. Let us consider another limitation. Suppose that the boundary B of the domain D becomes extremely irregular, e.g., strongly wiggling. Then again the bounding constants will become large, because we shall have to let the number of cells v realizing localization grow without any limit in order to satisfy the conditions that were imposed on the local canonical transformations.

This is really an intrinsic limitation and not just a technical difficulty. It has to do with the type of canonical transformations that can be used. Such limitations were already met by Fefferman[3] Presumably, it has much to do with the result by Van Hove[6] showing that some apparently benign classical canonical transformations can throw our functions out of the initial Hilbert space, i.e., that they do not correspond to a unitary transformation.

5. Systems where the classical equations of motion tranform well-behaved cells in phase space into very irregular regions have been actively investigated under the name of K-systems.[7] This remark provides a natural link with another fundamental problem already encountered in paper I: irreversibility.

6. IRREVERSIBILITY AND LOGIC

Irreversibility was encountered in paper I when discussing measurement theory. In fact, most if not all the measuring instruments for quantum systems must behave irreversibly, if only to work.

The question of irreversibility has so many aspects that only a few of

them will be mentioned here and in fact the only ones that will be found useful so that a solution might appear.

First recall the example of the classical description of a system of N particles using the BBGKY hierarchy. One can define a distribution function $f_N(x, p)$ for the N particles and define from it by integration one-particle f_1, two-particle f_2 distribution functions, etc. Each of them has an entropy

$$S_n = -k \int f_n \ln(f_n)\, dx_1 \cdots dx_n\, dp_1 \cdots dp_n$$

The Liouville theorem shows that S_N is a constant. As for S_1, it increases with time more rapidly than S_2, itself increasing more rapidly than S_3, and so on. Therefore a question arises: Is entropy only a measure of our insufficient knowledge of the system, and is its increase strictly associated with the mathematical representation we have used to describe it? If such were the case, thermodynamics would appear only as a branch of information theory.

The reverse question is then obviously: Can entropy be defined objectively? Such a question has much to do with the one asked about quantum mechanics, since one is already looking for an objective description of physical reality.

It will be useful ro recall also an interesting aspect of the description of irreversibility as it is done in information theory.[8] Assume that some collective variables q associated with quantum observables Q have been *chosen* to describe a quantum system. The initial values q^0 of q being given, one can find the density operator ρ_0 minimizing the information

$$I = \mathrm{Tr}(\rho_0 \ln \rho_0) \tag{6.1}$$

with prescribed values q_0 for the averages of the quantum operators Q.

Letting the density operator evolve according to the Schrödinger equation, i.e., using

$$\rho(t) = U(t)\, \rho_0 U^{-1}(t)$$

one meets the well-known difficulty that the entropy

$$S(t) = -k\, \mathrm{Tr}[\rho(t) \ln \rho(t)] \tag{6.2}$$

is found to be a constant. However, evolution takes $\rho(t)$ out of the family $\rho_m(q)$ that is defined by minimum information with a value q for the average of Q. Therefore, one projects the evolved density operator $\rho(\Delta t)$

after a time Δt upon this family by computing the average values of the collective variables Q at time Δt:

$$q(\Delta t) = \mathrm{Tr}[\rho(\Delta t) Q]$$

and redefining the new density operator $\rho_m[q(\Delta t)]$ as corresponding to these average values.

Accordingly, when seen from the standpoint of information theory, irreversibility looks like a succession of operations that take permanently the density operator back to the stable of wise ignorance.

My last remark will briefly deal with K-systems.[7] They are classical dynamical systems that are defined by the way their dynamical automorphism $g(t)$ acts on all the possible partitions of phase space into disjoint cells. Very roughly, one might say that for such a system, most cells become completely irregular under the action of $g(t)$ when time becomes large enough. Still very roughly, one can also consider an initial well-behaved cell C in phase space that has a volume V_0 and consider at a later time Δt the volume $V(\Delta t)$ of the convex envelope of the transformed cell $g(\Delta t) \cdot C$ that is some measure of what became of its extension and its meandering. For K-systems, one finds the relation

$$V(\Delta t) \sim V_0 \exp(\Sigma \, \Delta t) \tag{6.3}$$

where the coefficient Σ measures the rate of increase of the so-called Kolmogorov entropy that can be rigorously defined in terms of partitions. As far as I know, no clear-cut relation has been found between the Kolmogorov entropy and the physical entropy.

It is often suspected that the truly irreversible classical systems are in fact K-systems. However, we have just found that classical systems, not only dynamically but also logically, should be understood as having a quantum foundation. Then a difficult problem arises because almost nothing is known about quantum systems having for their classical version a K-system.[9]

I therefore introduce a new approach. According to what has been said in Section 5, the irregular domains that are produced by evolution when the classical automorphism $g(t)$ of a K-system acts on well-behaved cells will rather soon escape the conditions where Theorems 1 and 2 are valid. Although there may be a classical representation of logic describing the system at some given time by using well-behaved cells, soon afterward the evolved cells will no longer correspond to projectors and will not satisfy the compatibility conditions.

Thus, one arrives immediately at what appears to be the root of the problem: Since Planck's constant is finite and therefore quantum mechanics

should enter the physical description of reality, the evolution of K-systems breaks the correspondence between the quantum and the classical representations of logic. Stated very harschly, this means that K-systems are precisely those where *classical dynamics has no reliable logical meaning.*

When making such a statement, I do not suggest of course that the mathematical analysis of classical K-systems is wrong, but that it has no direct correspondence with the interpretable behavior of the system.

Such a point of view completely modifies the methodological framework that one may use in elaborating the theory of quantum K-systems. It leads naturally to another approach to the problem of irreversibility that will now briefly be described. Its present form is still rather primitive and its should be considered at best, for the time being, as a program for future research.

7. IRREVERSIBILITY AND DYNAMICS

This approach is the following. Assume that we have an objectively defined family of partitions of phase space in cells, say $\{C_\alpha^j\}$, where C_α^j is the cell number α for one of the allowable partitions denoted itself by the index j. When saying that it is objectively defined, I mean that such a family is obeying some well-defined mathematical criterion and that it is not something completely at our disposal. In fact, such families do exist. They have been found by Fefferman as the partitions that obey the following property[3]: Each cell C_α can be brought upon a rectangular cell by a well-behaved canonical transformation so that to each of them corresponds a good projector Φ_α according to Theorem 1, and therefore a subspace $\mathscr{H}_\alpha$ of $\mathscr{H}$. Taking $\mathscr{H}$ as the direct sum $\sum_\alpha \mathscr{H}_\alpha$ means that the cells can be chosen in such a way that the Hamiltonian be approximately diagonalized. More precisely, the deep and general result obtained by Fefferman is that the Hamiltonian H appears essentially in that representation in a block-diagonal form, nondiagonal terms being small in norm.

Fefferman's construction of these cells is highly involved, but I shall need only three basic facts concerning them:

(i) There is in general a large family of such partitions.

(ii) They are curved cells that are obtained from a rectangular cell by an allowable canonical transformation. The definition of such a transformation by Fefferman relies on the character of its generating function as a pseudo-differential symbol.

(iii) The partitions are such that classical mechanics is a good approximation of quantum mechanics up to *relative* errors of a given order ε. In practice, Fefferman's ε is what I called Δ in Section 4.

It might seem that our freedom of choice for describing the system is still rather large, since the choice of ε is at our disposal and that we are therefore far from objectivity and still in the domain of arbitrariness. However, the power of Δ exhibited in Eq. (4.14) shows that there may be a *best* Δ such that terms like R_1 and R_2 in Eq. (4.18) are comparable, in which case the ultimate limit of a precise classical description of physics has been reached.

Choosing Δ in this way, the families $\{C_\alpha^j\}$ are all equivalent. To each partition of phase space j is associated a partition of unity $\{\phi_\alpha^j\}$ where each C^∞ function $\phi_\alpha^j(x, p)$ has C_α^j for its essential support and where

$$\sum_\alpha \phi_\alpha^j(x, p) = 1 \tag{7.1}$$

whatever may be the point (x, p). Let Φ_α^j be the corresponding semi-projector operators and let us from now on work with one fixed partition, omitting therefore the index j.

Let the initial density operator of the system be a given operator ρ. Let also $\rho(x, p)$ denote the associated W-symbol, where one can recognize the Wigner distribution function[5] in phase space. Let us decompose it along the partition of unity (7.1) in the following form:

$$\rho(x, p) = \sum_\alpha \rho_\alpha(x, p) \equiv \sum_\alpha [\rho(x, p)\, \phi_\alpha(x, p)] \tag{7.2}$$

Such a decomposition corresponds to what Fefferman calls the pseudo-differential calculus associated with the Hamiltonian H. It can deal in principle with any reasonable kind of density operator.

Using Eq. (2.4), I now define the numbers

$$g_\alpha = \mathrm{Tr}(\rho_\alpha \Phi_\alpha) = \int \rho_\alpha(x, p)\, dx\, dp\, h^{-N} \tag{7.3}$$

They are positive as being the integrals of a Wigner function over a sufficiently large cell. Furthermore, because of Eq. (2.4), one has

$$\sum_\alpha g_\alpha = 1 \tag{7.4}$$

since $\mathrm{Tr}\, \rho = 1$. Then, calling again $[C_\alpha]$ the volume of cell C_α, one can consider a function $f(x, p)$ that represents the "best" description of the system by a classical distribution function. It is defined as a function taking the constant value $g_\alpha [C_\alpha]^{-1}$ over each of the cells C_α. According to Eq. (7.3), it satisfies the normalization property of a probability distribution, i.e.,

$$\int f(x, p)\, dx\, dp = 1 \tag{7.5}$$

Our problem will now be to find a master equation controlling the time evolution of this distribution function.

If the classical version of the dynamical system is a K-system, the cells will become uncontrollably deformed after a small interval of time that remains, however, determined by classical evolution. Such a situation has much in common with what was met above when the evolution was described in the framework of information theory and I now adopt a similar technique.

Let us admit that the small details of the functions $\rho_\alpha(x, p)$, except for their average values f_α in cells C_α, are not accessible. The opposite assumption would violate Fefferman's description of classical mechanics to order Δ. Then all the classical propositions that can be meaningful must be limited to cells no small than the C_α and to distribution functions of the same type as the function f I have introduced. This is equivalent to using density operators whose W-symbols are given by

$$\rho_m(x, p) = \sum_\alpha f_\alpha \phi_\alpha(x, p) \tag{7.6}$$

or $\rho_m = \sum_\alpha f_\alpha \Phi_\alpha$.

The system having the K-property, the time-transformed operator $U(t)\Phi_\alpha U^{-1}(t)$, will no longer approximate a projector after a finite time of order Σ^{-1}, Σ as given in Eq. (6.3). So let Δt be a time smaller than Σ^{-1} and such that $U(\Delta t)\Phi_\alpha U^{-1}(\Delta t)$ is still almost a projector. Then it is possible to project it back on the initial projectors by writing

$$U(\Delta t)\Phi_\alpha U^{-1}(\Delta t) = \left(\sum_\beta \Phi_\beta\right)[U(t)\Phi_\alpha U^{-1}(\Delta t)]\left(\sum_\gamma \Phi_\gamma\right)$$

More directly, one may consider the W-symbol $\psi_\alpha(x, p)$ associated with the operator $U(\Delta t)\Phi_\alpha U^{-1}(\Delta t)$ and decompose it along our partition of unity according to

$$\psi_\alpha(x, p) = \sum_\beta \psi_\alpha(x, p)\,\phi_\alpha(x, p) \approx \sum_\beta \Theta_{\alpha\beta}\phi_\beta(x, p) \tag{7.7}$$

where

$$\Theta_{\alpha\beta} = [C_\alpha]^{-1} \int \psi_\alpha \phi_\beta \, dx\, dp$$

Then the representation (7.5) of the density operator becomes

$$\rho_m(x, p, \Delta t) = \sum_\alpha f_\alpha(\Delta t)\,\phi_\alpha(x, p) \tag{7.8}$$

with

$$f_\alpha(\Delta t) = \sum_\beta \Theta_{\alpha\beta} f_\beta \tag{7.9}$$

As a result, one has obtained the following master equation for the evolution of the distribution function:

$$\Delta f_\alpha / \Delta t = \sum_\beta \Theta_{\alpha\beta} f_\beta \tag{7.10}$$

It has the form of a Markov process, for which the conditions of approach to equilibrium have been rather well investigated.

This can be considered as the end of the program. Were it to be fulfilled, its obvious interest would be that it involves in principle no arbitrariness, so that the entropy

$$S = -\sum_\alpha f_\alpha \ln(f_\alpha) \tag{7.11}$$

should have an objective meaning. In principle, no more detailed classical form of entropy could be defined that would respect the limitations that are imposed on the classical representation of logic by quantum mechanics.

If this approach turns out to be correct, it would imply that irreversibility holds (at least?) when the classical version of the dynamical system, or of a part of its is of *K*-type. Thus, it would provide a link between the Kolmogorov entropy and the physical entropy through the coefficient Σ that controls the rate in the master equation and therefore the rate of increase of the physical entropy.

Maybe this might go deeper: Irreversibility would emerge in physical systems (as opposed to systems described mathematically as classical systems, although such a description might not have a physical meaning), because of two complementary properties: Looking at them as classical systems, they are *K*-systems, but looking at them as an approximation to quantum physics, there is a limit to the precision of the approximation that breaks under evolution. In other words: classical physics, being nonlinear, can lead to arbitrarily wild sorts of evolution; quantum physics looks nicer because it is a linear theory, so that their correspondence can only be kept for a short time. Thus, irreversibility might appear as a bridge taking its foundations on both the classical and the quantum worlds and could even not appear when treated strictly in classical mechanics *or* in quantum mechanics. Such a bridge could not even be thought of in the Copenhagen version of quantum mechanics, although of course other approaches to

irreversibility have been found efficient in its context (see in particular Pauli[10]). The fact that such a bridge has become possible warrants further investigation.

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